Double Series

Def: Let f be a double sequences and let be the double sequences defined by the equation

$$S(p,q) = \sum_{k=1}^{p} \sum_{n=1}^{q} f(m,n) \Rightarrow m,n^{th} \text{ partial Sum}$$

The pair (f,s) is called a double Series and is denoted by the symbol $\sum_{m,n} f(m,n)$ or more breifly by $\sum f(m,n)$

The double series is said to converge to the sum a if $\lim_{p,q\to\infty} S(p,q) = a$

Each number $f(m,n)$ is called a term of the double Series and each $S(p,q)$ is a partial sum.

If $\sum f(m,n)$ has only positive terms it is easy to show that it converges if and only if of partial sums is bounded.

We say, $\sum f(m,n)$ converges absolutely of $\sum |f(m,n)|$ converges.

Thm 8.18 is valid for duble series

Rearangement Theorem For Double Series:

Def: Let f be a double Sequence and let g be a one to one function defined on Z^+ with rang $Z^+ \times Z^+$

Let G be the sequence defined by

$$G(n) = f(g(n)), \forall n \in Z^{-1}$$

Then g is said to be an arrangement of the double sequence defined by f into the Sequence G.

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Theorem : 8.42

State and prove rearrangement theorem of double series.

Statement :

Let $\sum f(m,n)$ be a given double series and let g be a arrangement of the double sequence f into a sequence G.

a) $\sum G(n)$ converges absolutely if and only if $\sum f(m,n)$ converges absolutely.

Assume that $\sum f(m,n)$ converge absolutely, with sum S we have further

b) $\sum G(n) = S$

c) $\sum_{n=1}^{\infty} f(m,n)$ and $\sum_{m=1}^{\infty} f(m,n)$ both converge absolutely

d) If $A_m = \sum_{n=1}^{\infty} f(m,n)$ and $B_n = \sum_{m=1}^{\infty} f(m,n)$, both series sum S. that is

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = S$$

proof:

Given that : (i) $\sum f(m,n)$ is a given double series

(ii) g is an arrangement of the double sequence f into a sequence G.

(ie) $g : z^{+} \longrightarrow z^{+} \times z^{+}$ is a 1-1 and onto function defined as, $G(n) = f(g(n)) \;\forall\; n \in z^{+} \longrightarrow (1)$

To prove : (a)

Suppose that $\sum (G(n))$ converges absolutely

(ie) $\sum |G(n)|$ converges

TPT : $\sum f(m,n)$ converges absolutely

(ie) $\sum |f(m,n)|$ converges

We know that $\sum |f(m,n)|$ converges iff the set of partial sums is bounded.

Let $S(p,q) = \sum_{m=1}^{p} \sum_{n=1}^{q} |f(m,n)|$

TPT: $S(p,q)$ is bounded

Since each term $|f(m,n)|$ in $S(p,q)$ is a term of the series $\sum |G(n)|$ we have,

$$S(p,q) = \sum_{m=1}^{p} \sum_{n=1}^{q} |f(m,n)|$$

$$\leq \sum |G(n)|$$

Since $\sum |G(n)|$ is convergent, $S(p,q)$ is bounded.

$\therefore$ $\sum |f(m,n)|$ has bounded partial sums.

$\sum |f(m,n)|$ is convergent

$\Rightarrow$ $\sum f(m,n)$ converges absolutely.

Conversely,

Assume that $\sum f(m,n)$ converges, absolutely

ie) $\sum |f(m,n)|$ converges

TPT: $\sum G(n)$ converges absolutely

ie) $\sum |G(n)|$ converges

Let T_k denote the k^{th} partial sum of $\sum |G(n)|$

ie) $T_k = |G(1)| + |G(2)| + \cdots + |G(k)|$

$= |f(g(1))| + |f(g(2))| + \cdots + |f(g(k))|$

$\leq \sum |f(m,n)|$

Since $\sum |f(m,n)|$ is convergent, T_k is bounded.

$\sum |G(n)|$ has bounded partial sums.

$\therefore$ $\sum |G(n)|$ converges.

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ii) $\sum G(n)$ converges absolutely

b) Assume that $\sum f(m,n)$ converges absolutely with sum S

To prove : (b)

T.P.T : $\sum_{n=1}^{\infty} G(n) = S$

Since $\sum f(m,n)$ converges absolutely by part (a) $\sum G(n)$ converges absolutely

(i.e) $\sum |G(n)|$ and $\sum G(n)$ converges

we shall now p.T $\sum_{n=1}^{\infty} G(n) = S$

For every arrangement g we can constructed a sequence G from the double sequence 'f'

we will S.T the sum of the series $\sum G(n)$ is independent of the function g used to construct G from 'f'.

To prove this let 'h' be another arrangement of the double sequence 'f' into a sequence H

Then we have

$$G(n) = f(g(n)) \text{ and } H(n) = f(h(n)) \longrightarrow ②$$

Since $h : z^+ \longrightarrow z^+ \times z^+$ is 1-1 and onto h^{-1} exists

and $h^{-1} : z^+ \times z^+ \longrightarrow z^+$ is 1-1 and onto

Let $k : z^+ \longrightarrow z^+$ be defined by,

$$k(n) = h^{-1}(g(n)) \ \forall \ n \in z^+$$

Then $hok = g$ and hence

$$\begin{aligned} H(k(n)) &= f(h(k(n))) \quad [\text{by } ②] \\ &= f(h(h^{-1}(g(n)))) \\ &= f(g(n)) \\ &= G(n) \quad [\text{by } ②] \end{aligned}$$

Since k is a 1-1 mapping of Z^+ onto Z^+, the series $\sum H(n)$ is a rearrangement of $\sum G(n)$

" Since every rearrangement of an absolutely convergent and has the same sum"

$\sum H(n)$ is convergent and has the same sum $\sum G(n)$

i.e) $\sum H(n) = \sum G(n)$

Let us denote the common sum by S' (we will) show that $S' = S$

For this purpose.

$$\text{Let} \quad S(p,q) = \sum_{m=1}^{p} \sum_{n=1}^{q} |f(m,n)|$$

$$\lim_{p,q \to \infty} S(p,q)$$

$\therefore$ For any given $\epsilon > 0$ we can find N, so that

$$0 \leq T - S(p,q) < \epsilon/2 \quad \text{whenever} \quad p \geq N \text{ and } q \geq N \longrightarrow (2)$$

Now consider the partial sum of the series $\sum G(n)$ and $\sum f(m,n)$

$$\text{Let} \quad t_k = \sum_{n=1}^{k} G(n)$$

$$\text{and} \quad S(p,q) = \sum_{m=1}^{p} \sum_{n=1}^{q} f(m,n)$$

Since every term in the double series $\sum f(m,n)$ is in the series $\sum G(n)$, we can find sufficiently large M such that,

t_M includes all terms $f(m,n)$ with

$$1 \leq m \leq N+1 \qquad 1 \leq n \leq N+1$$

Then it follows that,

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$t_{nn} - s(N+1, N+1)$ is a sum of terms $f(m,n)$ in

with either $m > N+1$ or $n > N+1$

i.e.) $t_{nn} - s(N+1, N+1) = \sum_{\substack{n > N+1 \\ \text{or} \\ m > N+1}} f(m,n)$

There fore if $n \geq M$ we have,

$$|t_n - s(N+1, N+1)| \leq \sum_{m,n > N+1} |f(m,n)|$$

$$= T - S(N+1, N+1)$$

$$< \epsilon/2$$

$\therefore$ If $n \geq M$ then

$$|t_n - s(N+1, N+1)| < \epsilon/2 \longrightarrow (4)$$

Since $S = \lim_{p,q \to \infty} s(p,q)$

it follows that

$S - s(N+1, N+1)$ is a sum of terms $f(m,n)$

with either $m > N+1$ or $n > N+1$

there we have,

$$|S - s(N+1, N+1)| \leq T - S(N+1, N+1) < \epsilon/2$$

$$|S - s(N+1, N+1)| < \epsilon/2 \longrightarrow (5)$$

From (4) and (5) we get find,

$$|t_n - S| = |t_n - s(N+1, N+1) + s(N+1, N+1) - S|$$

$$\leq |t_n - s(N+1), N+1)| + |s(N+1, N+1) - S|$$

whenever $n \geq N$

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

Thus given $\epsilon > 0$, there exists M such that

$$|t_n - S| < \epsilon \quad \text{whenever} \quad n \geq M$$

this shows that $\lim_{n \to \infty} t_n = S$

But $\lim_{n \to \infty} t_n = \sum G(n) = S'$ and hence $S = S'$

Thus we have $\sum G(n) = S$

This proves (b)

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To prove : c

Assume that $\sum f(m,n)$ converges absolutely with sum S

To prove : $\sum_{n=1}^{\infty} f(m,n)$ and $\sum_{m=1}^{\infty} f(m,n)$ both converge absolutely

For each $n \in N$ the single series $\sum_{m=1}^{\infty} f(m,n)$ a a subseries of $\sum G(n)$

Since $\sum f(m,n)$ is a sub series of $\sum G(n)$, $\sum_{m=1}^{\infty} f(m,n)$ converges absolutely (by thm 8.35)

To prove (d):

Assume that $\sum f(m,n)$ converges absolutely with sum S.

To prove: If $A_m = \sum_{n=1}^{\infty} f(m,n)$ and $B_n = \sum_{m=1}^{\infty} f(m,n)$

Then, both series $\sum A_m$ and $\sum B_n$ converge absolutely and both have sum S

Since $\sum f(m,n)$ converges absolutely with sum S.

$$A_m = \sum_{n=1}^{\infty} f(m,n) \text{ and } B_n = \sum_{m=1}^{\infty} f(m,n)$$

both the series $\sum A_m$ and $\sum B_n$ converges absolutely and both have sum S thus we have,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n) = S$$

NOTE: The series $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n)$ and $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n)$ are called iterated series converge of both iterated series does not imply their equality, for example suppose

$$f(m,n) = \begin{cases} 1 & \text{if } m = n+1, \; n = 1, 2 \ldots \\ -1 & \text{if } m = n-1, \; n = 1, 2 \ldots \\ 0 & \text{otherwise} \end{cases}$$

then $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n) = -1$

but $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n) = 1$

A Sufficient Condition For Equality Of Iterated Series:

Theorem 8.43

Let f be a complex valued double sequence assume that $\sum_{n=1}^{\infty} f(m,n)$ converges absolutely for each fixed m and that

$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |f(m,n)|$ converges then

a) The double series $\sum_{m,n=1}^{\infty} f(m,n)$ converges absolutely

b) The series $\sum_{m=1}^{\infty} f(m,n)$ converges absolutely in each 'n'

c) Both iterated series $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n)$ and $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n)$ converge absolutely and we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n) = \sum_{m,n=1}^{\infty} f(m,n)$$

proof: Given

(i) f is a complex valued double sequence

(ii) $\sum_{n=1}^{\infty} f(m,n)$ converges absolutely.

for each fixed m and then

$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |f(m,n)|$ converges.

To prove (a) i.e.) TPT the double series $\sum_{m,n} f(m,n)$ converges absolutely

Let g be an arrangement of the double sequence f into a sequence G

Then $G(n) = f(g(n))$, $n \in Z^+$

Consider the partial sum of $\sum |G(n)|$

$$\sum_{n=1}^{k} |G(n)| = \sum_{n=1}^{\infty} |f(g(n))|$$

$$\leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |f(m,n)|$$

Since $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |f(m,n)|$ converges, the partial sums of $\sum |G(n)|$ are bounded

$\Rightarrow \sum |G(n)|$ converges

$\Rightarrow \sum G(n)$ converges absolutely.

$\Rightarrow$ The double Series $\sum_{m,n} f(m,n)$ converges absolutely

This proves (a) [by thm. 8.42 (a)]

TP: (b)

(e) T.P.T $\sum_{m=1}^{\infty} f(m,n)$ converges absolutely for each n

In part (a) we have prove that the double series $\sum_{m,n} f(m,n)$ converges absolutely.

Then $\sum G(n)$ converges absolutely (by part (a) thm 8.42)

Now for each $n \in N$ the single series $\sum_{m=1}^{\infty} f(m,n)$ is a subseries of $\sum G(n)$.

Since $\sum_{m=1}^{\infty} f(m,n)$ is a subseries of $\sum G(n)$ and Since $\sum G(n)$ converges absolutely $\sum_{m=1}^{\infty} f(m,n)$ converges absolutely.

Toprove: C

To prove both iterated Series $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n)$ and $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n)$ converges absolutely

Let $A_m = \sum_{n=1}^{\infty} f(m,n)$ and $B_n = \sum_{m=1}^{\infty} f(m,n)$

Since $\sum f(m,n)$ converges absolutely with sum S, both the Series $\sum_{m=1}^{\infty} A_m$ and $\sum_{n=1}^{\infty} B_n$ converges absolutely and both have sum S.

Thus we have,

$$\sum_{m=1}^{\infty} A_m = \sum_{n=1}^{\infty} B_n = S$$

$$\Rightarrow \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = S.$$

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Theorem: 8.44

Let $\sum a_m$ and $\sum b_n$ be two absolutely convergent series with sums A and B respectively. Let f be the double sequence defined by the double ~~eq~~ equation

$$f(m,n) = a_m b_n \quad \text{if } (m,n) \in Z^+ \times Z^+$$

Then $\sum_{m,n} f(m,n)$ converges absolutely and has the sum AB.

Proof: Given,

(i) $\sum a_m$ and $\sum b_n$ are two absolutely convergent series with sums A and B respectively.

(ii) f is a double sequence defined by

$$f(m,n) = a_m b_n \quad \text{if } (m,n) \in Z^+ \times Z^+$$

TPT: $\sum_{m,n} f(m,n)$ converges absolutely and has sum AB

Consider $\sum_{m=1}^{\infty} |a_m| \sum_{n=1}^{\infty} |b_n|$

$$= \left(|a_1| + |a_2| + \cdots\right) \sum_{n=1}^{\infty} |b_n|$$

$$= |a_1| \sum_{n=1}^{\infty} |b_n| + |a_2| \sum_{n=1}^{\infty} |b_n| + \cdots$$

$$= \sum_{m=1}^{\infty} \left(|a_m| \sum_{n=1}^{\infty} |b_n| \right)$$

$$= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} |a_m||b_n| \right)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_m||b_n|$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |f(m,n)| \longrightarrow (1)$$

Since $\sum a_m$ and $\sum b_n$ converges absolutely and has sum A and B respectively

$$\sum_{m=1}^{\infty} |a_m| = A \quad \text{and} \quad \sum_{n=1}^{\infty} |b_n| = B$$

$$(1) \Rightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |f(m,n)| = AB$$

IIIly, we can P.T $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |f(m,n)| = AB$

we have $\sum_{m,n} f(m,n) = \sum_{m,n} a_m b_n$ converges absolutely

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